

QCD_{1+1} in the Limit of a Large Number of Colors and Flavors

Michael Engelhardt

Department of Condensed Matter Physics

Weizmann Institute of Science

Rehovot 76100, Israel

()

QCD in 1+1 dimensions is examined in the limit of a large number of colors and flavors. The Hamiltonian matrix is given in a Fock space spanned by 't Hooft meson states and, for the case of zero fermion mass, a submatrix is diagonalized numerically to give the low-lying spectrum as a function of N_F/N_C . Pair creation effects generate bound states which are complicated mixtures of components of different meson number. There are a number of nontrivial zero modes; in the massive part of the spectrum some states tend to a well-defined $N_F \gg N_C$ limit while others become unstable and disappear. The masses of most states remain remarkably constant over a large range of N_F/N_C .

I. INTRODUCTION

Since its inception, the large N_C expansion [1] [2] has afforded a host of valuable insights into QCD dynamics beyond standard perturbation theory in the coupling constant. Besides providing a solvable example of a nonabelian gauge theory in the case of one space dimension [2], it has been significant e.g. in clarifying the relation between the Skyrme model and QCD and in qualitatively explaining the Zweig rule [3]. On the other hand, the number of colors N_C is not the only parameter of QCD which can be regarded as large. The same procedure may also be applied to the number of flavors N_F . In the real world, especially at higher energies, where the heavy quarks contribute to the dynamics, the effective N_F/N_C ratio becomes larger than unity. Thus it seems natural to investigate in what way the presence of a large number of flavors modifies the large N_C picture and how those parts of large N_C phenomenology which appear to be successes survive the modification. Early work in this direction was done by Veneziano [4] and some more recent explorations into the field of multiflavor gauge theory have been carried out e.g. in connection with bosonization treatments [5] [6].

Particularly the low-dimensional models such as QCD_{1+1} with a large number of colors become considerably simplified in the framework of light-cone quantization [2]. At this point it should be remarked that the light-cone treatment corresponds in some respects to an effective theory with well-defined limitations, as was elucidated in [7]. In the work presented here, no attention will be paid to such questions. The results will be obtained entirely within the usual naive light-cone framework and no firm claim is made as far as their validity in normal coordinates is concerned. The prevailing experience gives rise to the hope that the light-cone treatment does yield correct excitation spectra.

The present foray into the multiflavor world focuses purely on the mesonic spectrum of the model with zero quark mass matrix. The Hamiltonian is calculated in a (truncated) basis of states made up of 't Hooft mesons, which diagonalizes the Hamiltonian for $N_F = 1$, $N_C \rightarrow \infty$. Increasing the number of flavors such that N_F/N_C remains finite as $N_C \rightarrow \infty$ induces nonvanishing interactions between the 't Hooft mesons; the evolution of the low-lying spectrum as a function of N_F/N_C can be followed by rediagonalizing the Hamiltonian. Here, only the flavor singlet sector is discussed; however, the generalization to finite flavor density presents no conceptual problem. Since the numerical work up to now was carried out with limited computational resources, only the one-meson and two-meson (i.e. two- and four-parton) sectors could be included.

II. THE MESONIC HAMILTONIAN

The Hamiltonian of multiflavor QCD_{1+1} in the light-cone gauge with zero quark mass matrix reads in momentum space

$$\begin{aligned}
 H = & -\frac{g^2}{4\pi} \frac{N_C^2 - 1}{N_C} \sum_{a,i} \int \frac{dk}{k} : q_{ai}^\dagger(k) q_{ai}(k) : \\
 & -\frac{g^2}{8\pi} \sum_{a,b,i,j} \int \frac{dq}{q^2} dk dk' \left(: q_{ai}^\dagger(k) q_{bi}(k' + q) q_{bj}^\dagger(k') q_{aj}(k - q) : \right.
 \end{aligned} \tag{1}$$

$$+ \frac{1}{N_C} : q_{ai}^\dagger(k) q_{ai}(k-q) q_{bj}^\dagger(k') q_{bj}(k'+q) : \Big)$$

Here, i and j are the color indices, a and b the flavor indices. Note that the fermion fields do not carry a Dirac index, since one of the components completely decouples in the light-cone gauge. The Hamiltonian has already been normal-ordered, by applying Wick's theorem, with respect to the vacuum of the theory, which is the perturbative one:

$$q_{ai}^\dagger(p)|0\rangle = 0 \text{ for } p < 0 \quad (2)$$

$$q_{ai}(p)|0\rangle = 0 \text{ for } p > 0 \quad (3)$$

The singularities in the integrands finally are defined via the principal value prescription. As already mentioned in the introduction, all subtleties associated with the singular nature of the light-cone description are disregarded here; the reader is referred to [7].

The Hamiltonian (1) conserves baryon number, total momentum, and overall color. Here, the focus will be on the color singlet sector [10] with zero baryon number and total momentum denoted by $2K$. Due to the confining Coulomb potential, the physical spectrum consists entirely of color singlet quark-antiquark bound states, i.e. mesons, and conglomerates thereof [11] [12]. It thus seems meaningful to consider the Hamiltonian matrix in a Fock space spanned by color singlet mesonic excitations above the (perturbative) vacuum. Mesons are created by the bilinear operators

$$M_{Qnab}^\dagger = \frac{1}{\sqrt{N_C}} \int_0^Q dp f_n(Q, p) \sum_i q_{ai}^\dagger(p) q_{bi}(p-Q) \quad (4)$$

where the f_n are a complete set of bound state wave functions. A particularly suitable such set are 't Hooft's wave functions,

$$f_n(Q, p) = \frac{1}{\sqrt{Q}} \phi_n(p/Q) \quad (5)$$

where the ϕ_n satisfy the equation

$$-\left(\frac{1}{x} + \frac{1}{1-x}\right) \phi_n(x) - \int \frac{dy}{(x-y)^2} \phi_n(y) = \frac{\mu_n^2}{m_0^2} \phi_n(x) \quad (6)$$

with the boundary condition

$$\phi_n'(0) = \phi_n'(1) = 0 \quad (7)$$

The mass scale m_0 is given by

$$m_0^2 = \frac{g^2}{2\pi} \frac{N_C^2 - 1}{N_C} \quad (8)$$

and μ_n^2 is the invariant mass squared of the n -th 't Hooft meson. Apart from the exact zero-mode solution $\phi_0(x) = 1$, a suitable orthonormal set in which to expand the ϕ_n in order to solve (6) numerically is given in this case by cosine functions.

With this choice, the Hamiltonian is already diagonal in the basis defined by the M_{Qnab}^\dagger in the case that N_F remains finite and $N_C \rightarrow \infty$. Note that the operators M^\dagger are not suitable for a rigorous bosonization of the theory since they only obey canonical bosonic commutation relations up to additional terms of the order $1/N_C$. It is now a straightforward, albeit lengthy, procedure to calculate the Hamiltonian matrix by commuting through quark operators. One obtains

$$\langle 2K, n, a, b | H | 2K, m, c, d \rangle = \delta_{ac} \delta_{bd} \delta_{nm} \frac{\mu_n^2}{4K} \quad (9)$$

$$\begin{aligned} \langle 2K, n, a, b | H | K + Q/2, m, c, d; K - Q/2, m', c', d' \rangle = \\ \frac{m_0^2}{2} \frac{1}{\sqrt{N_C}} \frac{1}{(2K)^{3/2}} \left[\delta_{c'd} \delta_{ac} \delta_{bd'} f_{mm'n} \left(\frac{K + Q/2}{2K} \right) + \delta_{cd'} \delta_{ac'} \delta_{bd} f_{m'mn} \left(\frac{K - Q/2}{2K} \right) \right] \end{aligned} \quad (10)$$

$$\langle K + Q/2, n, a, b; K - Q/2, n', a', b' | H | K + Q'/2, m, c, d; K - Q'/2, m', c', d' \rangle =$$

$$\begin{aligned}
& \delta_{ac}\delta_{bd}\delta_{a'c'}\delta_{b'd'}\delta_{nm}\delta_{n'm'}\delta(Q-Q')\left(\frac{\mu_n^2}{2K+Q}+\frac{\mu_{n'}^2}{2K-Q}\right)+\frac{m_0^2}{8K^2N_C}\times \\
& \left[\delta_{a'c}\delta_{ac'}\delta_{bd}\delta_{b'd'}f_{mm'nn'}^{EXC}\left(\frac{K+Q/2}{K+Q'/2},\frac{K-Q/2}{K+Q'/2}\right)+\delta_{ac}\delta_{a'c'}\delta_{b'd}\delta_{bd'}f_{m'mnn'}^{EXC}\left(\frac{K+Q/2}{K-Q'/2},\frac{K-Q/2}{K-Q'/2}\right)\right. \\
& +\delta_{a'c'}\delta_{ab'}\delta_{bd}\delta_{cd'}f_{mm'nn'}^{ANN}\left(\frac{K+Q/2}{K+Q'/2},\frac{K-Q/2}{K+Q'/2}\right)+\delta_{ac'}\delta_{a'b}\delta_{b'd}\delta_{cd'}f_{mm'n'n}^{ANN}\left(\frac{K-Q/2}{K+Q'/2},\frac{K+Q/2}{K+Q'/2}\right) \\
& \left.+\delta_{a'c}\delta_{ab'}\delta_{bd'}\delta_{c'd}f_{m'mnn'}^{ANN}\left(\frac{K+Q/2}{K-Q'/2},\frac{K-Q/2}{K-Q'/2}\right)+\delta_{ac}\delta_{a'b}\delta_{b'd'}\delta_{c'd}f_{m'mnn'n}^{ANN}\left(\frac{K-Q/2}{K-Q'/2},\frac{K+Q/2}{K-Q'/2}\right)\right]
\end{aligned} \tag{11}$$

with the form factors (note that the ϕ_n are defined to be zero outside the interval $[0, 1]$)

$$\begin{aligned}
f_{mm'n}(v) &= \frac{1}{\sqrt{v(1-v)}} \int_0^v dx \int_0^{1-v} dy \phi_m(x/v) \phi_{m'}(y/(1-v)) \frac{\phi_n(x) - \phi_n(v+y)}{(v+y-x)^2} \\
f_{mm'nn'}^{ANN}(v, w) &= \frac{(v+w)^2}{\sqrt{vw(v+w-1)}} \int_0^v dx \int_0^w dy \phi_n(x/v) \phi_{n'}(y/w) \frac{\phi_m(1+x-v) \phi_{m'}(y/(v+w-1))}{(w+x-y)^2} \\
f_{mm'nn'}^{EXC}(v, w) &= \frac{(v+w)^2}{\sqrt{vw(v+w-1)}} \int_0^v dx \int_0^w dy \phi_n(x/v) \phi_{n'}(y/w) \times \\
& \quad \frac{(\phi_m(1+x-v) - \phi_m(y))(\phi_{m'}(x/(v+w-1)) - \phi_{m'}((v+y-1)/(v+w-1)))}{(v-1+y-x)^2}
\end{aligned} \tag{12}$$

Here, in the kinetic part of the two-meson-two-meson matrix element, a convention of definite ordering of the momenta in the states was assumed, namely $Q, Q' \geq 0$. This avoids double-counting two-meson states. Also, in this kinetic part, $1/N_C$ -suppressed terms in the normalization of the states were neglected. All interaction parts, on the other hand, are valid for any Q, Q' and no terms were neglected.

The processes described by $f_{mm'n}$, $f_{mm'nn'}^{ANN}$, and $f_{mm'nn'}^{EXC}$ are associated with a flavor flow represented diagrammatically in graphs (A),(B) and (C) of figure (1), respectively. These graphs clarify the large N_F counting associated with each vertex; this will crucially influence which contributions survive in the limit N_F/N_C constant, $N_C \rightarrow \infty$.

Note that the Hamiltonian can only change the number of mesons by one. To change it by two, it would have to contain a contribution consisting of four parton creation or annihilation operators with respect to the vacuum; this, however, would completely commute through any meson operators to the right or left, respectively, ultimately annihilating the vacuum in relation to which the Hamiltonian is already normal-ordered. Equations (9)-(11) completely define the Hamiltonian in the subspace containing up to two mesons. But for the fact that the meson operators M^\dagger only obey canonical commutation relations up to terms of order $1/N_C$, the three-meson and four-meson interactions described by (9)-(11) would already represent the full dynamics in mesonic language. This is because the Coulomb interaction, as is clear from the original quark representation (1), can only modify at most two out of an arbitrarily long string of meson operators as it is being commuted through to eventually annihilate the vacuum. However, the $1/N_C$ -suppressed additional terms in the meson commutators introduce additional interactions in larger clusters which in general must be treated on the same footing as the vertices derived above. Since the Hilbert space will here be truncated to the one- and two-meson sectors, these terms need not be considered further in the present calculation.

III. THE FLAVOR-SINGLET SECTOR

The above considerations still apply to an arbitrary structure in flavor space; the results will now be specialized to the flavor singlet sector. A flavor singlet meson can be constructed by the superposition

$$|Q, n\rangle_S = \frac{1}{\sqrt{N_F}} \sum_a M_{Qnaa}^\dagger \tag{13}$$

whereas a flavor singlet two-meson state may consist either of two singlet mesons or of two mesons carrying flavor spin one coupled to a singlet:

$$|Q, n; Q', n'\rangle_{SS} = \frac{1}{N_F} \sum_{a,b} M_{Qnaa}^\dagger M_{Q'n'bb}^\dagger \tag{14}$$

$$|Q, n; Q', n'\rangle_{TT} = \frac{1}{N_F} \sum_{a,b} M_{Qnab}^\dagger M_{Q'n'ba}^\dagger \tag{15}$$

Carrying out the flavor algebra, the Hamiltonian matrix in these states takes the following form, where only terms which remain finite in the limit N_F/N_C finite, $N_C \rightarrow \infty$ are kept:

$${}_S\langle 2K, n | H | 2K, m \rangle_S = \delta_{nm} \frac{\mu^2}{4K} \quad (16)$$

$${}_S\langle 2K, n | H | K + \frac{Q}{2}, m; K - \frac{Q}{2}, m' \rangle_{SS} = 0 \quad (17)$$

$${}_S\langle 2K, n | H | K + \frac{Q}{2}, m; K - \frac{Q}{2}, m' \rangle_{TT} = \frac{m_0^2}{2} \sqrt{\frac{N_F}{N_C}} \frac{1}{(2K)^{3/2}} (1 + (-1)^{m+m'+n+1}) f_{mm'n} \left(\frac{K+Q/2}{2K} \right) \quad (18)$$

$${}_{SS}\langle K + \frac{Q}{2}, n; K - \frac{Q}{2}, n' | H | K + \frac{Q'}{2}, m; K - \frac{Q'}{2}, m' \rangle_{SS} = \delta_{nm} \delta_{n'm'} \delta(Q-Q') \left(\frac{\mu_n^2}{2K+Q} + \frac{\mu_{n'}^2}{2K-Q} \right) \quad (19)$$

$${}_{TT}\langle K + \frac{Q}{2}, n; K - \frac{Q}{2}, n' | H | K + \frac{Q'}{2}, m; K - \frac{Q'}{2}, m' \rangle_{SS} = \delta_{nm} \delta_{n'm'} \delta(Q-Q') \left(\frac{\mu_n^2}{2K+Q} + \frac{\mu_{n'}^2}{2K-Q} \right) \quad (20)$$

$${}_{TT}\langle K + \frac{Q}{2}, n; K - \frac{Q}{2}, n' | H | K + \frac{Q'}{2}, m; K - \frac{Q'}{2}, m' \rangle_{TT} = \delta_{nm} \delta_{n'm'} \delta(Q-Q') \left(\frac{\mu_n^2}{2K+Q} + \frac{\mu_{n'}^2}{2K-Q} \right) \\ + \frac{m_0^2}{8K^2} \frac{N_F}{N_C} (1 + (-1)^{m+m'+n+n'}) \left[f_{mm'n'n'}^{ANN} \left(\frac{K+Q/2}{K+Q'/2}, \frac{K-Q/2}{K+Q'/2} \right) + f_{mm'n'n}^{ANN} \left(\frac{K-Q/2}{K+Q'/2}, \frac{K+Q/2}{K+Q'/2} \right) \right] \quad (21)$$

Here, the symmetry behavior of 't Hooft's wave functions $\phi_n(x)$ around $x = 1/2$ was used, making explicit the selection rules for the meson excitation quantum numbers. Physically, this is the manifestation of CP invariance. These selection rules decouple states containing single-meson components of even excitation number from those containing single-meson components of odd excitation number. Note that this feature does not persist in the flavor non-singlet sector.

The Hamiltonian matrix obtained above exhibits in very transparent form the physics of the large N_F world: Flavor singlets do not interact among each other. Nonvanishing interactions only occur within flavor singlets, which resemble loops built out of alternating color strings and flavor "strings" in the sense that the two quarks at the end of a flavor string must be coupled to a singlet. Within these loops, color strings may break into two, creating a flavor string in between. Flavor strings are not necessarily binding; however, the diagonalization of the Hamiltonian will show that there are indeed bound states made up of more than one meson such that the flavor string effectively acts as a bond. Note the remarkable similarity of this picture to the model with one flavor, but adjoint color [8] [9].

In the original quark Hamiltonian, flavor does not appear as a quantum number related to the interaction; thus, the notion of a flavor string introduced above does not carry physical meaning in the same sense as the color string, which in fact represents a region of nonvanishing chromoelectric field energy density. The flavor string "force" is generated entirely by the quantum fluctuations, which manifest themselves in the diverging polarizability of the vacuum as $N_F \rightarrow \infty$. The weakness of the color forces between the mesons is offset by the ease with which quark-antiquark pairs are created. Formally, this is reflected in the fact that only graphs which imply the existence of free flavor indices to be summed over have a chance to compensate the usual $1/N_C$ -suppression of meson vertices by powers of N_F in the numerator. These are exactly the graphs in which a (flavor singlet) quark-antiquark pair is created or annihilated, i.e. a color string is broken into two or vice versa. Thus, only the graphs (A) and (B) in figure (1) survive; only the form factors $f_{mm'n}$ and $f_{mm'n'n'}^{ANN}$ enter the Hamiltonian matrix (16)-(21).

In contrast to the one-flavor theory, where pair creation is not enhanced, and which is simply solved by 't Hooft's mesons, the bound state spectrum will now consist of superpositions of states containing different numbers of mesons. In the case of the one-flavor, adjoint color theory [8] it was found that the low-lying spectrum is still very well described by a truncation to few quark-antiquark pairs. This gave rise to the hope that the restriction to one- and two-meson states imposed in the present calculation would also give an adequate description of the first few bound states. Ultimately, this point will have to be checked by enlarging the Hilbert space.

Numerical evaluation and diagonalization of the Hamiltonian matrix (16)-(21) gives the low-lying spectrum as a function of N_F/N_C . In the actual calculation, the two-meson states were supplied with a relative momentum wave function, i.e. the physical basis used was of the form (cf. the convention introduced further above of definite ordering of momenta in the kets):

$$\frac{1}{\sqrt{2K}} \int_0^{2K} dQ \psi_l(Q/2K) \frac{1}{\sqrt{2}} (|K+Q/2, m; K-Q/2, m' \rangle_{TT} \pm |K+Q/2, m'; K-Q/2, m \rangle_{TT}) \quad (22)$$

where the ψ_l were chosen to be (properly normalized) cosines (with the plus sign) and sines (with the minus sign). Note that for $m = m'$ only even (cosine) relative momentum wave functions are possible due to the Bose symmetry. Note also that the form factors (12) contain logarithmic divergences for zero quark mass which pose no problem if a continuous relative momentum wave function is introduced as above, whereas they make a direct discretization of the momentum difficult. The basis used in the computation consisted of eight one-meson and 26 two-meson states. Bound states were identified by subtracting the expectation value of the kinetic energy from the energy eigenvalue and checking for a negative result.

IV. NUMERICAL RESULTS

The numerical calculation yielded four nontrivial zero modes besides the zeroth 't Hooft meson $|2K, 0\rangle_S$, which is an eigenstate of the Hamiltonian for all N_F/N_C , as can be easily seen from the three-meson form factor $f_{mm'n}$ (cf. (12)). Furthermore, a spectrum of massive bound states, plotted in figure (2) as a function of N_F/N_C , was found. Some of the massive states disappear for intermediate values of N_F/N_C and reappear with very similar energy and mesonic composition. In such cases the range where the state disappears is bridged in figure (2) by the dotted lines to indicate the kinship between the spectral trajectories in different regions.

The different classes of states obtained can each be described in more detail as follows. The nontrivial zero modes are of the structure

$$\sum_{n \text{ odd}} a_n |2K, n\rangle_S + \frac{1}{\sqrt{K}} \int_0^{2K} dQ \chi(Q/2K) |K + Q/2, 0; K - Q/2, 0\rangle_{TT} \quad (23)$$

where the restriction to odd n comes simply from the selection rules already discussed above in connection with equations (16)-(21). It should be noted that the Hilbert space used in the present calculation contained the first four relative momentum states made up of two ground state 't Hooft mesons and also the first four single-meson states of odd excitation number. In this basis, four nontrivial zero modes were found. It thus seems natural to expect that the full theory in fact contains an infinity of zero modes of the form (23). For small N_F/N_C , the one-particle admixture vanishes as N_F/N_C . Such a scaling behavior is to be expected; for $N_F/N_C \rightarrow 0$, the Hamiltonian becomes diagonal in the mesonic basis and thus the eigenstates are dominated by a single (either one- or two-particle) Fock state, with N_F/N_C -suppressed admixtures obtainable in perturbation theory. The same behavior is exhibited by the binding energies. This feature of course also arises in the massive part of the spectrum to be discussed further below.

At $N_F = N_C$, the one-particle content ranges from 40% for states in which these one-particle admixtures are made up of low-lying 't Hooft mesons, downwards as more excited mesons participate. The binding energies at N_F/N_C are exactly $2m_0^2/K$ (up to the numerical accuracy of the calculation) for three of the zero modes, rising to $2.5m_0^2/K$ for the fourth. In the opposite limit of large N_F/N_C on the other hand, the two-meson admixture is suppressed as N_C/N_F and the states become mixtures of odd single 't Hooft meson states. As before, this scaling is easy to understand: For large N_F/N_C , the two-meson-two-meson interaction Hamiltonian, which is positive (this will be discussed further below), becomes dominant, and thus it is not favorable for a state to contain a high two-meson proportion. In order to maintain a meaningful balance between the different parts of the Hamiltonian, the two-meson amplitude must be proportional to $\sqrt{N_C/N_F}$, i.e. the two-meson content of the states must scale as N_C/N_F . Again, this property of course persists for the massive part of the spectrum. The binding energies of the zero modes must accordingly approach constants. Numerically one obtains two states with a binding energy of $8m_0^2/K$ for large N_F/N_C , one with binding energy $20m_0^2/K$, and one with $29m_0^2/K$. It has not been possible to give analytical solutions for the nontrivial zero modes; there must be subtle dynamical mechanism responsible for suppressing transitions from the single-meson admixtures in (23) to the massive part of the spectrum. It should be noted that some of the massive bound states to be discussed below also contain sizeable admixtures of odd single-meson states.

Continuing on to the massive part of the spectrum, the states containing odd single-meson components appear as the third and the fourth massive state in the $N_F/N_C \ll 1$ region (cf. figure (2)). In detail, these two states however exhibit some differences: The fourth state only exists at low N_F/N_C , and in the region shown consists of about equal parts of $|5\rangle$ and $|1, 1\rangle$ (the quantum numbers in the kets merely indicate the 't Hooft meson content, whereas the momentum quantum numbers have been suppressed). For very low N_F/N_C , though, the state becomes dominated by the $|1, 1\rangle$ part. By contrast, the third massive state becomes dominantly a one-meson state in the $N_F/N_C \rightarrow 0$ limit, namely a $|3\rangle$. As N_F/N_C rises, it acquires admixtures of $|0, 0\rangle$, $|1, 1\rangle$ and $|0, 2\rangle$ so that the $|3\rangle$ -component is still found with 55% probability at $N_F = N_C$. Whereas all other states found are dominated by two to three Fock components even in the transition region around $N_F = N_C$, this state becomes a strong mixture. This is especially pronounced where the state becomes unstable. Remarkably, as N_F/N_C rises beyond unity, the two-meson content rises further to reach 96% at $N_F/N_C = 50$; one third of this is the $|0, 0\rangle$ admixture. In accordance with the argument given further above, this ultimately upsets the balance between potential and kinetic energies. Thus, the state is very weakly bound above $N_F/N_C = 10$ and finally disappears around $N_F/N_C = 75$ instead of increasing its one-particle content to give a well-defined $N_F/N_C \rightarrow \infty$ limit. The large range over which this state survives is remarkable, though.

On the other hand, the rest of the massive states shown in figure (2) contain even single-meson states. Here again, one finds states which become dominated by a two-meson Fock component for $N_F/N_C \rightarrow 0$ (namely the second massive state, which becomes predominantly a $|0, 1\rangle$) and others dominated by a one-meson component in this limit: The first massive state becomes a $|2\rangle$, the fifth an $|8\rangle$. The one-particle fraction of the first massive state falls to about two thirds at $N_F = N_C$ due to admixtures of $|0, 1\rangle$ and then rises again to give a sensible $N_F/N_C \rightarrow \infty$ limit. Remarkably, in this region, its mass falls to zero. Apart from this last feature, the fifth massive state behaves very

similarly, acquiring a $|1, 2\rangle$ admixture of about one third in an intermediate regime of N_F/N_C and then becoming again a pure $|8\rangle$. The second massive state acquires a steadily rising admixture of $|4\rangle$ (about 25% at $N_F = N_C$) to rapidly become pure $|4\rangle$ for large N_F/N_C . The state which is predominantly a $|6\rangle$ for $N_F/N_C \rightarrow \infty$ only appears at $N_F/N_C > 10$; it is the fourth massive state in this region (cf. figure (2)). This behavior is most probably due to the truncation of Hilbert space, i.e. the state may be dominated at lower N_F/N_C by a two-meson component not included in the calculation.

Summing up, the gross features of the spectrum can be characterized as follows: On the one hand, there are the zero modes, which consist predominantly of $|0, 0\rangle$ with growing admixtures of odd single-meson states as N_F/N_C rises until these admixtures become dominant for large N_F/N_C . Concomitantly, the massive states containing odd single-meson components become unstable as N_F/N_C is increased, which is signalled by these states becoming strong mixtures without a dominant Fock component. On the other hand, the massive states containing even single-meson admixtures display a well-defined large N_F/N_C limit, where they are dominated by just these even single-meson states. The lowest of these states becomes massless for $N_F/N_C \rightarrow \infty$. The reason for this exceptional behavior is unclear; all other states display a remarkably stable mass over a wide range of N_F/N_C . Also this stability of the masses is a puzzle; the binding energies of the states on their own vary quite strongly, as is evident from the values indicated in figure (2).

V. COMMENTS AND OUTLOOK

Some comments are in order regarding the results presented above. The first of these concerns the low N_F/N_C limit. The reader will have noticed that not all of 't Hooft's mesons appear in the bound state spectrum in this limit. To be precise, only $|0\rangle$, $|2\rangle$, $|3\rangle$ and $|8\rangle$ are present; $|1\rangle$, $|4\rangle$, $|5\rangle$, $|6\rangle$ and $|7\rangle$ are missing. These states of course do appear in the numerical diagonalization, with small admixtures of two-meson states, but with a positive expectation value of the potential energy. Thus they are not bound and must rather be viewed as resonances. For very small N_F/N_C , this distinction becomes meaningless. An experimenter in 1+1 dimensions, working in a finite laboratory for a finite time, will observe all of 't Hooft's mesons as $N_F/N_C \rightarrow 0$. Strictly speaking, however, if for fixed N_F/N_C , the experimenter waits long enough, he/she will only find some of these mesons as stable particles. In the case of the higher mesons, the picture may be strongly influenced by the truncation of Hilbert space, but the absence of the $|1\rangle$ at least seems conspicuous. On the other hand, the bound states become very weakly bound for $N_F/N_C \rightarrow 0$, so that the experimenter would indeed observe the $|0, 1\rangle$ (this is the second massive state) as two essentially independent ground state and first excited 't Hooft mesons, respectively, in a finite experiment.

As a further comment, it should be stressed that, to generate bound states in the model, it is crucial to combine one-meson and two-meson states. Diagonalizing the two-meson sector on its own does not give binding, i.e. the two-meson–two-meson interaction matrix is positive, as mentioned above. This seems plausible, for the following reason: If there was a pure two-meson state with a negative expectation value of the interaction, then one would have a state with arbitrarily negative overall energy by simply choosing N_F/N_C suitably large such that the interaction dominates the kinetic part. On the other hand, the ground state (vacuum) energy of the theory has been defined to be zero, so that a state with negative energy should not be possible. Conversely, including one-meson states can very well generate binding through the three-meson vertex, and the energy is stabilized to remain positive by the two-meson–two-meson vertex; this is indeed precisely what happens, as mentioned above in the detailed discussion of the spectrum at large N_F/N_C . If one writes a mesonic Hamiltonian in terms of the operators M^\dagger and their adjoints such that it correctly reproduces the Hamiltonian matrix (9)-(11), it is the cubic term that may become negative, and the quartic term which stabilizes the theory.

Summing up, the large N_C , large N_F theory contains a considerably wealthier phenomenology than the one-flavor theory. It represents a more sophisticated model for mesons, incorporating pair creation effects and thus mimicking the presence of a meson cloud, thought to be an important ingredient of three-dimensional hadrons. Despite the complex behavior of the model, many of its features seem to be susceptible to numerical analysis, given the power of modern-day computers. In this respect, a significant improvement on the treatment presented here should be possible, since the latter only made use of two workstations. By using a mesonic basis, which already incorporates a sizeable part of the dynamics, the bulk of the computation lies in the evaluation of the Hamiltonian matrix rather than its diagonalization. The matrix elements, however, can be evaluated completely independent of each other, making a large improvement in speed possible by simply using parallel processors. Including sectors with a higher number of mesons demands the computation of higher-dimensional integrals; this will only be feasible beyond a certain point by employing Monte Carlo methods.

Besides improvements of the numerical aspects, there are a number of other extensions of this work which seem to merit attention. It would be interesting to understand in more detail the dynamics leading to binding. This endeavor

would profit greatly from some analytical understanding of the nontrivial zero modes. Especially the zero mode sector may to some extent be tractable analytically, at least in some approximation, since the form factors simplify considerably if some of the participating mesons are ground-state ones.

Furthermore, the approach followed here becomes the most questionable in the high N_F/N_C limit. There, the coupling between sectors of different particle number is the strongest and the inclusion of Fock components with three or more mesons could substantially alter the results. It would be helpful to devise an approximation scheme which is particularly suited for this potentially interesting new limit, i.e. one which makes explicit use of the smallness of the parameter N_C/N_F .

With a view to approaching more realistic theories, it may be helpful to work out the perturbative corrections in $1/N_C$. Finally, note also that the model is easily generalized to non-singlet flavor (flavor spin one objects will resemble open strings) as well as non-zero quark masses. In fact, in the latter case, the Hamiltonian matrix retains exactly the form (9)-(11); only the 't Hooft wave functions entering the form factors and the corresponding meson masses are modified. It is to be expected that quark masses suppress pair creation related effects. Thus, the physical spectrum should remain closer to the 't Hooft one for given N_F/N_C . The zero quark mass model considered in this work is the case furthest removed from the well-studied one-flavor theory, since it maximally allows for the pair creation processes which represent the essence of the model's nontrivial dynamics.

Acknowledgements

The author wishes to thank S.Levit and A.Schwimmer for motivating discussions. This work was supported by a MINERVA fellowship.

-
- [1] G.'t Hooft, Nucl. Phys. B72 (1974) 461
 - [2] G.'t Hooft, Nucl. Phys. B75 (1974) 461
 - [3] I.Klebanov, lectures presented at the workshop of the Graduiertenkolleg Erlangen-Regensburg (Germany) on "QCD and Hadron Structure", 9.6.-11.6.1992 at Kloster Banz; GK-Notes 3-92
 - [4] G.Veneziano, Nucl. Phys. B117 (1976) 519
 - [5] D.Gepner, Nucl. Phys. B252 (1985) 481
 - [6] G.D.Date, Y.Frishman and J.Sonnenschein, Nucl. Phys. B283 (1987) 365
 - [7] F.Lenz, M.Thies, K.Yazaki and S.Levit, Ann. Phys. (N.Y.) 208 (1991) 1
 - [8] K.Demeterfi, I.Klebanov and G.Bhanot, Nucl. Phys. B418 (1994) 15;
G.Bhanot, K.Demeterfi and I.Klebanov, Phys. Rev. D48 (1993) 4980
 - [9] D.Kutasov, Nucl. Phys. B414 (1994) 33
 - [10] M.Engelhardt and B.Schreiber, "Elimination of Color in 1+1-dimensional QCD", to appear in Z. Phys. A
 - [11] I.Bars and M.B.Green, Phys. Rev. D17 (1978) 537
 - [12] C.G.Callan, N.Coote and D.J.Gross, Phys. Rev. D13 (1976) 1649

FIG. 1. Flavor flow associated with the form factors $f_{mm'n}$, $f_{mm'nn'}^{ANN}$ and $f_{mm'nn'}^{EXC}$, respectively.

FIG. 2. Invariant square masses of the massive bound states as a function of N_F/N_C . Dotted lines are merely to guide the eye in connecting spectral trajectories of states which disappear at intermediate values of N_F/N_C . The figures indicate the binding energies of the states at selected points in the same units, i.e. $K\langle H_{pot} \rangle / 2m_0^2$.

This figure "fig1-1.png" is available in "png" format from:

<http://arXiv.org/ps/hep-ph/9412314v1>

This figure "fig1-2.png" is available in "png" format from:

<http://arXiv.org/ps/hep-ph/9412314v1>